Corrigendum: quasiconvex elastodynamics: weak-strong uniqueness for measure-valued solutions

Article (Accepted Version)

CORRIGENDUM: QUASICONVEX ELASTODYNAMICS: WEAK-STRONG UNIQUENESS FOR MEASURE-VALUED SOLUTIONS

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1. A Gårding-type inequality for quasiconvex functions

We correct a gap in the proof of Theorem 5.1 in [5]. Precisely, the proof results in the constant C_1 being dependent on t_0 in a way that cannot be controlled. In turn this implies that the constant C_1 in Proposition 4.3 also depends on t_0. Therefore, the constant C in the first floating inequality below (4.28) is time-dependent. This prevents the use of Grönwall's inequality which would conclude the proof of Theorem 4.1. In this corrigendum, we reprove Theorem 5.1 ensuring that the constants involved are time-independent. In doing so, we follow a strategy developed by Kristensen and Campos Cordero, see [1], and also Campos Cordero and Koumatos [2]. We also point that in [4] the crucial inequality (1.5) in the proof of Theorem 5.1 below has been obtained with a different proof.

We denote by

$$\mathcal{F}_K := \{H \in W^{1,\infty}(\overline{Q}, \mathbb{R}^{d\times d}) \,:\, \|H\|_{W^{1,\infty}} \leq K\},$$

noting that there exists a $K > 0$ such that the strong solution $\bar{F}(t,\cdot) \in \mathcal{F}_K$ for all times. We write $C(f, K)$ for a positive constant that depends only on the L^∞ bounds of a function f or any of its derivatives in a ball determined by K. Next, for $f : \mathbb{R}^{d\times d} \to \mathbb{R}$, we define the function $G_f : \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \to \mathbb{R}$ by

$$G_f(z,\xi) := f(z+\xi) - f(z) - Df(z) : \xi = \int_0^1 (1-s) D^2 f(z+s\xi)\xi : \xi \, ds. \tag{1.1}$$

We recall that the function $W \in C^2(\mathbb{R}^{d\times d})$ in [5] is required to be strongly quasiconvex with constant c_0, p-coercive with p-growth. Note that in the notation of [5], $G_W(\bar{F}(t,x),\xi) = G(t,x,\xi)$. We require a few preliminary results.

Lemma 1. *Let $W \in C^2(\mathbb{R}^{d\times d})$, strongly quasiconvex with constant c_0, p-coercive with p-growth. There exists a constant $c_2 = c_2(W,K)$ such that the function*

$$\tilde{W}(\xi) := W(\xi) - c_2|V(\xi)|^2$$

is p-coercive and satisfies the following:

(a) *There exists $C = C(\tilde{W}, K)$ such that for all $z \in \overline{B(0,K)}$, $\xi_1, \xi_2 \in \mathbb{R}^{d\times d}$*

$$|G_{\tilde{W}}(z,\xi_1) - G_{\tilde{W}}(z,\xi_2)| \leq C(|\xi_1| + |\xi_2| + |\xi_1|^{p-1} + |\xi_2|^{p-1})|\xi_1 - \xi_2|.$$

(b) *$\tilde{W}$ is strongly quasiconvex with constant $c_0/2$ at all $\xi \in \overline{B(0,K)}$, i.e.*

$$\int_Q \tilde{W}(\xi + \nabla\varphi) - \tilde{W}(\xi) \geq \frac{c_0}{2}\int_Q |V(\nabla\varphi)|^2, \quad \forall\, |\xi| \leq K, \forall\, \varphi \in W^{1,p}(Q).$$

(c) *For all $\xi \in \overline{B(0,K)}$ and all $Q' \subset Q$ it holds that*

$$\int_{Q'} D^2\tilde{W}(\xi)\nabla\varphi : \nabla\varphi \geq c_0 \int_{Q'} |\nabla\varphi|^2 \quad \forall\, \varphi \in W_0^{1,p}(Q'). \tag{1.2}$$

Proof. For (a) see [2, Lemma 4.4]. Part (b) follows by applying (a) to the strongly convex $f(\xi) = |V(\xi)|^2$ in place of $\tilde{W}$ and (c) by viewing quasiconvexity as a minimality condition and considering the second variation. □

The following result is inspired by Dafermos [3, Lemma 4.3].

Proposition 1. *There exists $c_1 = c_1(W, K) > 0$ such that for any $H \in \mathcal{F}_K$*

$$\int_Q D^2\tilde{W}(H(x))\nabla\varphi : \nabla\varphi \geq \frac{c_0}{2}\int_Q |\nabla\varphi|^2 - c_1\int_Q |\varphi|^2, \quad \forall\varphi \in W^{1,2}(Q).$$

Proof. Fix $\delta > 0$ and a finite cover $\{Q_i\} \subset Q$, $Q_i = Q_i(x_i, r_i)$, such that

$$|D^2\tilde{W}(H(x)) - D^2\tilde{W}(H(x_i))| \leq c_0\delta(1-\delta)^2.$$

Since $H \in \mathcal{F}_K$ and $\tilde{W} \in C^2(\mathbb{R}^{d\times d})$, the cover can be chosen uniformly for $H \in \mathcal{F}_K$. Next, choose a partition of unity $\{\rho_i\}$ subordinate to the cover $\{Q_i\}$ such that $\operatorname{supp}\rho_i \subset Q_i$ and $\sum_i \rho_i^2 = 1$. Given $\varphi \in W^{1,2}(Q)$, we find that for all $H \in \mathcal{F}_K$,

$$\begin{aligned}\int_Q D^2\tilde{W}(H(x))\nabla\varphi : \nabla\varphi &= \sum_i \int_{Q_i} \rho_i^2 D^2\tilde{W}(H(x_i))\nabla\varphi : \nabla\varphi \\ &\quad + \sum_i \int_{Q_i} \rho_i^2 \left[D^2\tilde{W}(H(x)) - D^2\tilde{W}(H(x_i))\right]\nabla\varphi : \nabla\varphi \\ &\geq \sum_i \int_{Q_i} D^2\tilde{W}(H(x_i))(\rho_i\nabla\varphi) : (\rho_i\nabla\varphi) \\ &\quad - c_0\delta(1-\delta)^2\int_Q |\nabla\varphi|^2. \end{aligned} \tag{1.3}$$

Note that $\rho_i\nabla\varphi = \nabla(\rho_i\varphi) - \varphi\otimes\nabla\rho_i$ with $\rho_i\varphi \in W_0^{1,2}(Q_i)$ and $|H(x_i)| \leq K$. Then, by (1.2) and Young's inequality, we infer that

$$\int_{Q_i} \rho_i^2 D^2\tilde{W}(H(x_i))\nabla\varphi : \nabla\varphi \geq c_0(1-\delta)\int_{Q_i} |\nabla(\rho_i\varphi)|^2 - C\int_{Q_i}|\varphi|^2, \tag{1.4}$$

where $C = C(\tilde{W}, K, \delta)$. Through Young's inequality we also find that

$$\int_{Q_i} \rho_i^2 D^2\tilde{W}(H(x_i))\nabla\varphi : \nabla\varphi \geq c_0(1-\delta)^2\int_{Q_i} \rho_i^2|\nabla\varphi|^2 - C(\delta)\int_{Q_i}|\varphi|^2,$$

where $C(\delta)$ also depends on $\|\nabla\rho_i\|_\infty$, in turn depending only on δ and W. Then, after summing up, (1.3) results in

$$\int_Q D^2\tilde{W}(H(x))\nabla\varphi : \nabla\varphi \geq c_0(1-\delta)^3\int_Q |\nabla\varphi|^2 - C(\delta)\int_Q|\varphi|^2.$$

To conclude the proof, fix $\delta = 1 - 2^{-1/3}$ and rename $C = C(W, K) =: c_1$. □

We next present a Proposition which is used repeatedly.

Proposition 2. *Let $(H_k) \subset \mathcal{F}_K$, $(h_k) \subset W^{1,p}(Q)$, $(a_k) \subset \mathbb{R}$ such that*

$$a_k^{-1}V(h_k) \to 0 \text{ strongly in } L^2(Q), \quad \left(a_k^{-1}V(\nabla h_k)\right) \text{ is bounded in } L^2(Q).$$

Then,

$$\liminf_{k\to\infty} \frac{c_0}{4}a_k^{-2}\int_Q |V(\nabla h_k)|^2 \leq \liminf_{k\to\infty} a_k^{-2}\int_Q G_{\tilde{W}}(H_k(x), \nabla h_k).$$

Proof. The proof is identical to [2, Proposition 4.6], noting that there is no dependence on the lower order terms (h_k) and no assumptions on (h_k) are required. □

Lastly, we present a decomposition lemma whose proof can be found in [1].

Proposition 3. *Let $\psi_k \rightharpoonup \psi$ in $H_0^1(Q)$. Suppose that $(\eta_k) \subset (0,1]$ and $(\eta_k\psi_k)$ is bounded in $W^{1,p}(Q)$. Then, there exist $g_k \in C_c^\infty(Q)$ and $b_k \in H^1(Q)$ such that*

(a) $\psi_k = \psi + g_k + b_k$;

(b) g_k, $b_k \rightharpoonup 0$ *in* $W^{1,2}(Q)$ *and* $\eta_k g_k$, $\eta_k b_k \rightharpoonup 0$ *in* $W^{1,p}(Q)$;
(c) $\nabla b_k \to 0$ *in measure;*
(d) $\big(|\nabla g_k|^2\big)$ *and* $(|\eta_k \nabla g_k|^p)$ *are equiintegrable.*

We immediately infer Theorem 5.1 in [5].

Proof of Theorem 5.1 in [5]. We show that there exist constants $C_0 = C_0(W,K)$ and $C_1 = C_1(W,K)$ such that for all $H \in \mathcal{F}_K$ and all $\varphi \in W^{1,p}(Q) \cap H^1_0(Q)$,

$$\int_Q |V(\nabla\varphi)|^2 dx \le C_1 \int_Q G_W(H(x), \nabla\varphi) dx + C_0 \int_Q |V(\varphi)|^2 dx. \tag{1.5}$$

Then, choose $H = \bar{F}(t_0,\cdot) = \nabla\bar{y}(t_0,\cdot) \in \mathcal{F}_K$ for some $K > 0$ uniform in t_0, and $\varphi = z^k(t,\cdot) - \bar{y}(t_0,\cdot)$ where z_k is constructed in Lemma 5.6 in [5]. Next, integrate in time and take the limit $k \to \infty$, using the equiintegrability of $(|\nabla z^k|^p)$ and that (∇z^k) generates $(\nu^F_{t_0,x})_{x\in Q}$, to conclude the proof of Theorem 5.1, i.e. that

$$\begin{aligned}\int_Q &\langle \nu^F_{t_0,x}, |V(\xi - \bar{F}(t_0,x))|^2\rangle dx \\ &\le C_1 \int_Q \langle \nu_{t_0,x}, G_W(\bar{F}(t_0,x), \xi - \bar{F}(t_0,x))\rangle \, dx + C_0 \int_Q |V(y(t_0,x) - \bar{y}(t_0,x))|^2 dx.\end{aligned}$$

In order to show (1.5), by Proposition 4 below, there exists $\varepsilon_0 > 0$ such that

$$\int_Q G_{\tilde{W}}(H(x), \nabla\varphi) + \frac{c_1}{2}|\varphi|^2 \ge 0 \tag{1.6}$$

whenever $\|\varphi\|_{L^p(Q)} < \varepsilon_0$. Then, by the definition of $\tilde{W}$ and the strong convexity of $f(\xi) = |V(\xi)|^2$, (1.6) says that whenever $\|\varphi\|_{L^p(Q)} < \varepsilon_0$,

$$C(K)\int_Q |V(\nabla\varphi)|^2 \le C(K)\int_Q G_f(H,\nabla\varphi) \le \int_Q G_W(H,\nabla\varphi) + \frac{c_1}{2}|V(\varphi)|^2.$$

Then, we can conclude (1.5) as for $\|\varphi\|_{L^p} \ge \varepsilon_0$, by the coercivity of W and Young's inequality, it holds that

$$\begin{aligned}\int_Q G_W(H(x),\nabla\varphi) &\ge \int_Q c|H + \nabla\varphi|^p - C(W,K) - C(\delta)|DW(H)|^q - \delta|\nabla\varphi|^p \\ &\ge -C(W,K) + \tilde{c}\int_Q |\nabla\varphi|^p \ge -\frac{C(W,K)}{\varepsilon_0^p}\int_Q |\varphi|^p + \tilde{c}\int_Q |\nabla\varphi|^p,\end{aligned} \tag{1.7}$$

for δ small enough. This concludes the proof after noting that $\|V(\nabla\varphi)\|^2_{L^2} \le 1 + 2\|\nabla\varphi\|^p_{L^p}$ and that, by Poincaré's inequality, $\varepsilon_0^p \le C\|\nabla\varphi\|^p_{L^p}$. □

We are thus left to prove the proposition below which is the core of the argument.

Proposition 4. *There exists $\varepsilon_0 > 0$ such that for all $H \in \mathcal{F}_K$ and all $\varphi \in W^{1,p}(Q) \cap H^1_0(Q)$ with $\|\varphi\|_{L^p(Q)} < \varepsilon_0$ it holds that*

$$\int_Q G_{\tilde{W}}(H(x), \nabla\varphi) + \frac{c_1}{2}|\varphi|^2 \ge 0. \tag{1.8}$$

Proof. To prove (1.8), we proceed by contradiction. Suppose (1.8) is false. Then we can find $(H_k) \subset \mathcal{F}_K$, $H \in \mathcal{F}_K$, and $(\varphi_k) \subset W^{1,p}(Q) \cap H^1_0(Q)$ such that $\|\varphi_k\|_{L^p(Q)} \to 0$, $H_k \overset{*}{\rightharpoonup} H$ in $W^{1,\infty}(Q)$ and

$$\int_Q G_{\tilde{W}}(H_k(x), \nabla\varphi_k(x)) + \frac{c_1}{2}|\varphi_k(x)|^2 < 0. \tag{1.9}$$

Step 1: We show that $\varphi_k \to 0$ in $W^{1,p}(Q)$ and that

$$\sup_k \frac{\beta_k^p}{\alpha_k^2} =: \Lambda < \infty, \text{ where } \alpha_k = \|\nabla\varphi_k\|_{L^2(Q)}, \ \beta_k = \|\nabla\varphi_k\|_{L^p(Q)}. \tag{1.10}$$

By (1.9), after using the p-coercivity of $\tilde{W}$ and Young's inequality, we find that $(\nabla\varphi_k)$ is bounded in $W^{1,p}(Q)$. We may thus apply Proposition 2 with $a_k = 1$ and $h_k = \varphi_k$ to find that, by (1.9),

$$\liminf_{k\to\infty} \frac{c_0}{4}\int_Q |V(\nabla\varphi_k)|^2 \le \liminf_{k\to\infty}\int_Q G_{\tilde{W}}(H_k, \nabla\varphi_k) \le 0,$$

and $\varphi_k \to 0$ in $W^{1,p}(Q)$. Regarding (1.10), the p-coercivity of $\tilde{W}$ implies that

$$\int_Q G_{\tilde{W}}(H(x), \nabla\varphi_k) \ge d\int_Q |\nabla\varphi_k|^p - c\int_Q |\nabla\varphi_k|^2, \tag{1.11}$$

as discussed in [6, S.3.2], where $d = d(\tilde{W}, K)$, $c = c(\tilde{W}, K)$. Then (1.10) follows after dividing by α_k^2 and noting (1.9).

Step 2: Let $\psi_k := \alpha_k^{-1}\varphi_k$. Since $\|\nabla\psi_k\|_{L^2(Q)} = 1$ and $\psi_k \in H_0^1(Q)$, we find that $\psi_k \rightharpoonup \psi$ in $W^{1,2}(Q)$. Moreover, setting $\eta_k = \frac{\alpha_k}{\beta_k} \in (0,1]$, we have that $(\eta_k\psi_k)$ is bounded in $W^{1,p}(Q)$. We may thus decompose ψ_k to find $g_k \in C_c^\infty(Q)$, $b_k \in H^1(Q)$ as in Proposition 3. Write

$$f_k(x) = \alpha_k^{-2}\left[G_{\tilde{W}}(H_k, \alpha_k\nabla\psi_k) - G_{\tilde{W}}(H_k, \alpha_k\nabla b_k)\right] \tag{1.12}$$

and note that, since $\alpha_k\psi_k = \varphi_k$, by (1.9),

$$\int_Q f_k(x) + \alpha_k^{-2}G_{\tilde{W}}(H_k, \alpha_k\nabla b_k) + \frac{c_1}{2}|\psi_k|^2 < 0. \tag{1.13}$$

Apply Proposition 2 with $a_k = \alpha_k$ and $h_k = \alpha_k b_k$ to the term $\alpha_k^{-2}G_{\tilde{W}}(H_k, \alpha_k\nabla b_k)$. By using Step 1 we get

$$\frac{c_1}{2}\int_Q |\psi|^2 + \liminf_{k\to\infty}\int_Q f_k(x) \le 0. \tag{1.14}$$

Step 3: Let $\nu = (\nu_x)_{x\in Q}$ be the $W^{1,2}$ gradient Young measure generated by the sequence ψ_k and recall that $H_k \to H$ in $C^0(Q)$. We show that

$$\frac{1}{2}\int_Q \langle \nu_x, D^2\tilde{W}(H(x))\xi : \xi\rangle \le \liminf_{k\to\infty}\int_Q f_k(x). \tag{1.15}$$

In particular, in conjunction with (1.14), we infer that

$$\frac{1}{2}\int_Q c_1|\psi|^2 + \langle \nu_x, D^2\tilde{W}(H(x))\xi : \xi\rangle \le 0. \tag{1.16}$$

To prove (1.15) we show the equiintegrability of (f_k) defined in (1.12). Indeed, by Lemma 1 (a) and for a constant $C = C(\tilde{W}, K)$, Young's inequality gives

$$\begin{aligned}
|f_k| &\le C(|\nabla\psi_k| + |\nabla b_k| + \alpha_k^{p-2}|\nabla\psi_k|^{p-1} + \alpha_k^{p-2}|\nabla b_k|^{p-1})|\nabla\psi_k - \nabla b_k| \\
&\le \delta C(|\nabla\psi_k|^2 + |\nabla b_k|^2) + C(\delta)|\nabla(\psi + g_k)|^2 \\
&\quad + \delta C(\alpha_k^{p-2}|\nabla\psi_k|^p + \alpha_k^{p-2}|\nabla b_k|^p) + C(\delta)\alpha_k^{p-2}|\nabla(\psi + g_k)|^p,
\end{aligned}$$

recalling that, by Proposition 3, $\nabla\psi_k - \nabla b_k = \nabla(\psi + g_k)$. However, by Proposition 3, ψ_k and b_k are bounded in $W^{1,2}(Q)$, whereas $(|\nabla(\psi + g_k)|^2)$ is equiintegrable. Similarly, since $\alpha_k^{p-2} = \Lambda\eta_k^p$ we infer that $\alpha_k^{p-2}|\nabla\psi_k|^p$ and $\alpha_k^{p-2}|\nabla b_k|^p$ are bounded, whereas $\alpha_k^{p-2}|\nabla(\psi + g_k)|^p$ is equiintegrable. Hence, (f_k) is also equiintegrable and for $\varepsilon > 0$ fixed, we can find m_ε such that

$$\int_Q f_k > -\varepsilon + \int_{\{|\nabla\psi_k| < m\}\cap\{|\nabla b_k| < m\}} f_k, \quad \forall\, m \ge m_\varepsilon. \tag{1.17}$$

This follows as $\nabla b_k \to 0$ in measure and $\lim_{r\to\infty} \sup_k\{|\nabla\psi_k| > r\} = 0$. Also, since $\int_Q \langle \nu_x, |\xi|^2\rangle < \infty$, we may assume that for all $m \geq m_\varepsilon$,

$$\int_Q \langle \nu_x, D^2\tilde{W}(H)\xi : \xi\rangle = \int_Q \langle \nu_x, D^2\tilde{W}(H)\xi : \xi\chi_{B(0,m)}(\xi)\rangle + \varepsilon, \tag{1.18}$$

where χ_A denotes the indicator function of a set $A \subset \mathbb{R}^{d\times d}$. Since $B(0,m)$ is open, for all $x \in Q$ the function $\xi \mapsto D^2\tilde{W}(H(x))\xi : \xi\chi_{B(0,m)(\xi)}$ is lower semicontinuous and, as $(\nabla\psi_k)$ generates $(\nu_x)_{x\in Q}$ and $H_k \to H$ in $C^0(Q)$, we deduce that

$$\begin{aligned}\int_Q \langle \nu_x, D^2\tilde{W}(H)\xi : \xi\chi_{B(0,m)}(\xi)\rangle &\leq \liminf_{k\to\infty} \int_{\{|\nabla\psi_k|<m\}} D^2\tilde{W}(H)\nabla\psi_k : \nabla\psi_k \\ &= \liminf_{k\to\infty} \int_{\{|\nabla\psi_k|<m\}} D^2\tilde{W}(H_k)\nabla\psi_k : \nabla\psi_k. \end{aligned}\tag{1.19}$$

Combining (1.19) with (1.18), we now infer that for all $m \geq m_\varepsilon$

$$\int_Q \langle \nu_x, D^2\tilde{W}(H)\xi : \xi\rangle \leq \liminf_{k\to\infty} \int_{\{|\nabla\psi_k|<m\}} D^2\tilde{W}(H_k)\nabla\psi_k : \nabla\psi_k + \varepsilon. \tag{1.20}$$

To conclude the proof, we next claim that

$$\frac{1}{2}\liminf_{k\to\infty} \int_{\{|\nabla\psi_k|<m\}} D^2\tilde{W}(H_k)\nabla\psi_k : \nabla\psi_k = \lim_{k\to\infty} \int_{\{|\nabla\psi_k|<m\}\cap\{|\nabla b_k|<m\}} f_k. \tag{1.21}$$

Before proving (1.21), note that in conjunction with (1.20) and (1.17) it says that

$$\frac{1}{2}\int_Q \langle \nu_x, D^2\tilde{W}(H(x))\xi : \xi\rangle \leq \liminf_{k\to\infty} \int_Q f_k + \frac{3\varepsilon}{2}.$$

By taking $\varepsilon \to 0$, (1.15) follows. To prove (1.21), set $A_k := \{|\nabla\psi_k| < m\}$ and $B_k := \{|\nabla b_k| < m\}$, so that

$$\begin{aligned}\chi_{A_k\cap B_k} f_k &= \chi_{A_k\cap B_k}\int_0^1 (1-s)\left[D^2\tilde{W}(H_k + s\alpha_k\nabla\psi_k) - D^2\tilde{W}(H_k)\right]\nabla\psi_k : \nabla\psi_k\, ds \\ &+ \chi_{A_k}\frac{1}{2}D^2\tilde{W}(H_k)\nabla\psi_k : \nabla\psi_k - \chi_{A_k}\frac{1}{2}D^2\tilde{W}(H_k)\nabla\psi_k : \nabla\psi_k\,(1-\chi_{B_k}) \\ &- \chi_{A_k\cap B_k}\int_0^1 (1-s)D^2\tilde{W}(H_k + s\alpha_k\nabla b_k)\nabla b_k : \nabla b_k\, ds =: I_1^k + I_2^k + I_3^k + I_4^k.\end{aligned}$$

Hence, it suffices to show that $I_i^k \to 0$, for $i = 1, 3, 4$, as $k \to \infty$ which follows by dominated convergence as $\alpha_k \to 0$, $H_k \to H$ in $C^0(Q)$ and $\nabla b_k \to 0$ in measure.

Step 4: We show how (1.16) combined with Proposition 1 leads to a contradiction . By (1.2), the function $\xi \mapsto D^2\tilde{W}(H(x))\xi : \xi$ is quasiconvex for each $x \in Q$. Since $(\nu_x)_{x\in Q}$ is a gradient Young measure, Jensen's inequality implies

$$\int_Q c_1|\psi|^2 + D^2\tilde{W}(H(x))\nabla\psi : \nabla\psi \leq \int_Q c_1|\psi|^2 + \langle \nu_x, D^2\tilde{W}(H(x))\xi : \xi\rangle \leq 0,$$

by (1.16), after adding $c_1|\psi|^2$ and integrating over Q. However, by Proposition 1,

$$\int_Q c_1|\psi|^2 + D^2\tilde{W}(\bar{F}(x))\nabla\psi : \nabla\psi \geq \frac{c_0}{2}\int_Q |\nabla\psi|^2, \quad \forall\psi \in W^{1,2}(Q),$$

i.e. $\nabla\psi = 0$ and, since $\psi \in H_0^1(Q)$, $\psi = 0$. We may thus apply Proposition 2 with $a_k = \alpha_k$ and $h_k = \alpha_k\psi_k$, recalling Step 1 to infer that

$$\begin{aligned} 0 < \frac{c_0}{4} \le \liminf_{k\to\infty} \frac{c_0}{4} \int_Q |\nabla\psi_k|^2 + \alpha_k^{p-2}|\nabla\psi_k|^p \le \liminf_{k\to\infty} \alpha_k^{-2} \int_Q G_{\tilde{W}}(H_k, \alpha_k\nabla\psi_k) \\ = \liminf_{k\to\infty} \alpha_k^{-2} \int_Q G_{\tilde{W}}(H_k, \nabla\varphi_k) + \frac{c_1}{2}|\varphi_k|^2 \le 0, \end{aligned}$$

by (1.9) as $\alpha_k^{-1}\varphi_k = \psi_k \to 0$. This contradiction concludes the proof. □

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